Thermal Entanglement between Alternate Qubits of a Four-qubit Heisenberg XX Chain in a Magnetic Field

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Abstract

The concurrence of two alternate qubits in a four-qubit Heisenberg XX chain is investigated when a uniform magnetic field B is included. It is found that there is no thermal entanglement between alternate qubits if B is close to zero. Magnetic field can induce entanglement in a certain range both for the antiferromagnetic and ferromagnetic cases. Near zero temperature, the entanglement undergoes two sudden changes with increasing value of the magnetic field B. This is due to the changes in the ground state. This novel property may be used as quantum entanglement switch. The anisotropy in the system can also induce the entanglement between two alternate qubits.

PACS#: 03.65.Ud, 03.67.Lx,75.10.Jm.

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The existence of entanglement shows many interesting properties in quantum systems. Its nonlocal quantum correlation has become one of the most valuable resources in quantum communication [1-3] and quantum computation [4, 5]. Recently, the concept of thermal entanglement in solids was introduced and studied in one-dimensional anisotropic Heisenberg model [6]. Thermal entanglement in two-qubit Heisenberg XX chain was investigated with and without an external magnetic field [7-9]. The entanglement between qubits of the next and next-next neighbors in an open spin chain and in multi-qubit Heisenberg model was presented [10-13].

In most of the previous investigations, the concurrence of two nearest-neighbor qubits is calculated as a measure of entanglement. For a pair of two qubits, the concurrence is given by [14,15]

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{1}$$

where the quantities $\lambda_i (i = 1, 2, 3, 4)$ are the square roots of the eigenvalues of the operator

$$\varrho = \rho_{12}(\sigma_1^y \otimes \sigma_2^y)\rho_{12}^*(\sigma_1^y \otimes \sigma_2^y) \tag{2}$$

in descending order. The values of the concurrence are ranged from zero to one when quantum states are changed for unentangled to maximally entangled states.

The state of the system at thermal equilibrium is represented by the density operator

$$\rho(T) = \frac{1}{Z}\exp(-\frac{H}{kT}) \tag{3}$$

where $Z = Tr[exp(-H/kT)]$ is the partition function, k is the Boltzmann's constant and T is the temperature. Since $\rho(T)$ represents a thermal state, the entanglement in the state is called thermal entanglement [16-17].

In this report, a Heisenberg XX model of four-qubit in a linear chain is investigated when a magnetic field B is included. The pairwise entanglement between alternate qubit is calculated. The four-qubit XXM Heisenberg model is described by the Hamiltonian

$$H_{XXM} = J\sum_{n=1}^{4}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + B\sum_{n=1}^{4}\sigma_n^z \tag{4}$$

where $\sigma_n^{\pm}$ are the raising and lowering operations, B is the external magnetic field and perpendicular to the chain, J is the strength of interaction. The value of positive and negative J corresponds to the antiferromagnetic and ferromagnetic cases respectively.

The eigenvalues and eigenstates of the Hamiltonian in Eq. (4) can be calculated analytically. In a four-qubit Heisenberg chain with periodic boundary conditions, the eigenvalues are given by

$$\begin{aligned}
&E_0 = -4B, && E_1 = 2J - 2B, && E_2 = E_4 = -2B \\
&E_3 = -2J - 2B, && E_5 = 2\sqrt{2}J, && E_6 = -2\sqrt{2}J \\
&E_7 = E_8 = E_9 = E_{10} = 0, && && E_{11} = 2J + 2B \\
&E_{12} = E_{14} = 2B, && E_{13} = -2J + 2B, && E_{15} = 4B
\end{aligned} \tag{5}$$

and the corresponding eigenstates can be explicitly expressed as

$$\begin{aligned}
|\psi_0\rangle &= |0000\rangle \\
|\psi_1\rangle &= \frac{1}{2}\left(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle\right) \\
|\psi_2\rangle &= \frac{1}{2}\left(|0001\rangle + i\,|0010\rangle - |0100\rangle - i\,|1000\rangle\right) \\
|\psi_3\rangle &= \frac{1}{2}\left(|0001\rangle - |0010\rangle + |0100\rangle - |1000\rangle\right) \\
|\psi_4\rangle &= \frac{1}{2}\left(|0001\rangle - i\,|0010\rangle - |0100\rangle + i\,|1000\rangle\right) \\
|\psi_5\rangle &= \frac{\sqrt{2}}{4}\left(|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle\right) + \frac{1}{2}\left(|0101\rangle + |1010\rangle\right) \\
|\psi_6\rangle &= \frac{\sqrt{2}}{4}\left(|0011\rangle + |0110\rangle + |1100\rangle + |1001\rangle\right) - \frac{1}{2}\left(|0101\rangle + |1010\rangle\right) \\
|\psi_7\rangle &= \frac{1}{2}\left(|0011\rangle + i\,|0110\rangle - |1100\rangle - i\,|1001\rangle\right) \\
|\psi_8\rangle &= \frac{1}{2}\left(|0011\rangle - |0110\rangle + |1100\rangle - |1001\rangle\right) \\
|\psi_9\rangle &= \frac{1}{\sqrt{2}}\left(|0101\rangle - |1010\rangle\right) \\
|\psi_{10}\rangle &= \frac{1}{2}\left(|0011\rangle - i\,|0110\rangle - |1100\rangle + i\,|1001\rangle\right) \\
|\psi_{11}\rangle &= \frac{1}{2}\left(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle\right) \\
|\psi_{12}\rangle &= \frac{1}{2}\left(|1110\rangle + i\,|1101\rangle - |1011\rangle - i\,|0111\rangle\right) \\
|\psi_{13}\rangle &= \frac{1}{2}\left(|1110\rangle - |1101\rangle + |1011\rangle - |0111\rangle\right) \\
|\psi_{14}\rangle &= \frac{1}{2}\left(|1110\rangle - i\,|1101\rangle - |1011\rangle + i\,|0111\rangle\right) \\
|\psi_{15}\rangle &= |1111\rangle
\end{aligned} \tag{6}$$

If the concurrence of two alternate qubits is considered, the reduced density matrix

$\rho_{13} = Tr_{24}\rho(T)$ can be given by

$$\rho_{13}(T) = \frac{1}{Z}\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & w & y & 0 \\ 0 & y & w & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \tag{7}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Where u, v, y are

$$\begin{aligned} u &= \frac{1}{2}\left(1 + e^{(2J+2B)\beta} + e^{-(2J-2B)\beta} + cosh 2\sqrt{2}J\beta\right) + e^{2B\beta} + e^{4B\beta} \\ v &= \frac{1}{2}\left(1 + e^{-(2J+2B)\beta} + e^{(2J-2B)\beta} + cosh 2\sqrt{2}J\beta\right) + e^{-2B\beta} + e^{-4B\beta} \\ y &= \frac{1}{2}\left(-1 + cosh\,(2J+2B)\,\beta + cosh\,(2J-2B)\,\beta + cosh 2\sqrt{2}J\beta\right) - cosh 2B\beta \end{aligned} \tag{8}$$

The partition function of the system is

$$Z = 4\,(1 + cosh 2B\beta) + 2\left[cosh 4B\beta + cosh\,(2J+2B)\,\beta + cosh\,(2J-2B)\,\beta + cosh 2\sqrt{2}J\beta\right] \tag{9}$$

with $\beta = \frac{1}{kT}$. In the following calculations k is set to be 1.0.

From the Eqs. (1), (2) and (7), the concurrence can be obtained

$$C = \frac{2}{Z}\max\left(|y| - \sqrt{uv}, 0\right) \tag{10}$$

Except the concurrence C of two qubits, the global entanglement Q of many-qubit pure states also needs to be considered. The global entanglement Q is introduced by [18-20]

$$Q = \frac{1}{N}\sum_{i=1}^{N} IC_i^2 \tag{11}$$

where the i-concurrence of IC_i means the entanglement between the qubit i and the other qubits and can be expressed as

$$IC_i = \sqrt{2[1 - Tr(\rho_i^2)]} \tag{12}$$

The concurrence C as functions of the magnetic field B and the temperature T is plotted in Fig. 1. The strength of interaction J is chosen to be 1.0. Fig. 1(a) is a

three-dimensional plot of the concurrence C as functions of B and T. From Fig. 1(a), it is clear that there is a two-peak structure in C. The two peaks appear symmetrically at two sides of B. There is no entanglement between alternate qubits at $B = 0$. The entanglement is also independent of T [10]. This can be understood since $B = 0$ the ground state will be $|\psi_6\rangle$, which is unentangled between alternate qubits. When the temperature $T = 0$, the ground state energy as a function of the magnetic field B is changed from E_6 to E_3, and then to E_0 for antiferromagnetic case. While the ground state energy is changed from E_5 to E_{11}, and then to E_{15} for ferromagnetic case. Therefore there should be a change in the concurrences C because of the change in the ground state. It can be seen that the increase of magnetic field B cannot induce entanglement when the temperature T is very high. When the temperature T is very low, the entanglement is increased with increasing value of the magnetic field $|B|$ to a maximum value. Then it is decreased and finally disappeared. The maximum value of the entanglement decreases when the temperature T is increased.

The contour map of the concurrence C as functions of the magnetic field B and the temperature T is plotted in Fig. 1(b). Four contours of $C = 0.5, 0.3, 0.1, 0$ are shown respectively. Beyond the contour $C = 0$, the entanglement is equal to zero. It means that there exists a critical temperature T_c. From the curve of $C = 0$ in Fig. 1(b), it can be seen that the critical temperature T_c depends on the magnetic field B. If $B < 0.09$, the concurrence C is always zero no matter the temperature T is increased or decreased. When $0.09 < B < 0.41$ or $B > 1.0$, there are two critical temperatures of T_c. When T is either above the lower part of the curve of $C = 0$ or below the upper part of $C = 0$, the concurrence C is always greater than zero. When $0.41 < B < 1.0$, there is a single value of the critical temperature T_c. When the temperature T is above the curve of $C = 0$, the entanglement is vanished no matter the magnetic field B is increased or decreased. For four-qubit nearest-neighbor XXM model, the critical temperature is independent of the magnetic field B [11].

The thermal entanglement of alternate qubits for $J = -1.0$ is also studied. The result is almost the same as that shown in Fig. 1. This means that the entanglement

exits for both antiferromagnetic and ferromagnetic cases.

The concurrence C, the global entanglement Q, and the i-concurrence IC_i are plotted in Fig. 2 as a function of the magnetic field B. The concurrence C of different qubits is plotted in Figs. 2(a) and 2(b) when the temperature T is varied with $T = 0.01$, 0.1, and 0.5. Fig. 2(a) is a plot of C for alternate qubits. From Fig. 2(a), it can be seen that the shape of C is like a square wave for low temperature of $T = 0.01$. The concurrence C keeps zero until B is increased to 0.41. Then the concurrence maintains a maximal value of $C = 0.5$ until it drops to zero at $B = 1.0$. In the limit of $T \to 0$, one has

$$\begin{aligned} &\lim_{T\to 0} C = 0 && |B| < (\sqrt{2}-1)|J| \\ &\lim_{T\to 0} C = \tfrac{1}{2} && (\sqrt{2}-1)|J| \le |B| \le |J| \\ &\lim_{T\to 0} C = 0 && |B| > |J| \end{aligned} \tag{13}$$

This can be understood as follows. When $|B| > |J|$ and $|B| < (\sqrt{2}-1)|J|$, the ground states are the unentangled state $|\psi_0\rangle$ and $|\psi_6\rangle$ respectively. While for $(\sqrt{2}-1)|J| < |B| < |J|$, the ground state is the maximally entangled state $|\psi_3\rangle$ of the two alternate qubits. When the temperature T is increased to 0.1, and 0.5, the shape of C is changed from square wave to a single peak. The maximal value of the concurrence C is decreased when T is increased. For finite temperatures, this definite ground state structure is smoothed out by the partition of higher states and therefore the concurrence C becomes smaller with higher temperatures. To compare this with that of the two nearest-neighbor qubits in the four-qubit XXM model, the concurrence C of the two nearest-neighbor qubits is plotted as a function of B in Fig. 2(b) with the same condition. For low temperature of $T = 0.01$, the entanglement keeps a constant value of $C = 0.46$ until it drops to a dip. It seems that the dip is due to the crossing of energy level at the point of $B = (\sqrt{2}-1)$ [11]. Then C maintains a maximal value of $C = 0.5$ until it drops to zero at $B = 1.0$. When the temperature T is increased to 0.1, the shape of C is changed to two peaks with almost the same dip. When T is increased to 0.5, the dip disappears. There is only a single peak at $B = 0$. From

Figs. 2(a) and 2(b), it can be seen that the effects of temperature on C is much stronger for alternate qubits than that for nearest qubits. When T is increased to 0.5, the curve of C in Fig. 2(b) is much higher than that in Fig. 2(a). It seems that the temperature affects the entanglement between weakly interacting alternate qubits stronger than that for strongly interacting nearest qubits. The global entanglement Q and the i-concurrence IC_i are plotted in Fig. 2(c). Due to the symmetry of the eigenstates in Eq. (6), the values of i-concurrence are the same. When $0 \leq B \leq 0.41$, $IC_1 = IC_2 = IC_3 = IC_4 = 1.0$. The value of i-concurrence is the same as that of Q. The two-qubit concurrence C is $C_{12} = C_{14} = C_{23} = C_{34} = (2\sqrt{2} - 1)/4$ and $C_{13} = C_{24} = 0$. These are shown in Figs. 2(a) and 2(b). Although the entanglement of three and four qubits cannot be discriminated, the additional entanglement of three and four qubits can be described by $1 - \sum C_{ij}^2 = 0.16$ [18, 19]. When $0.41 \leq B \leq 1.0$, $IC_1 = IC_2 = IC_3 = IC_4 = \sqrt{3}/2$. The global entanglement Q and the two-qubit concurrence C satisfy the relation of $Q = \frac{1}{2}\sum_{ij} C_{ij}^2$ [18, 19]. There is no additional entanglement of three and four qubits. When $B > 1.0$, there is no entanglement at all. All the values of C, Q and IC_i equal to zero. It is very interesting to note that in the low temperature limit the entanglement between two alternate qubits of the four-qubit XXM Heisenberg model undergoes two sudden changes when the magnetic field B is increased. This novel property may be used as quantum entanglement switch in quantum computing and quantum communications.

For a more general model of four-qubit Heisenberg XX chain, the anisotropic contribution needs to be considered. If the anisotropy is included, the Hamiltonian of the system can be written as

$$H_{XXZM} = H_{XXM} + \frac{J\Delta}{2}\sum_{n=1}^{4} \sigma_n^z \sigma_{n+1}^z \tag{14}$$

where Δ is the anisotropy parameter, H_{XXM} is given by Eq. (4). The system reduces to XX model when $\Delta = 0$ and the isotropic XXX model when $\Delta = 1.0$.

The concurrence C of alternate qubits is plotted in Fig. 3 as functions of the temperature T, the magnetic field B, and the anisotropic parameter Δ when the strength

of interaction J is 1.0. The concurrence C is plotted as functions of T and Δ in Fig. 3(a) when $B = 0.5$. From Fig. 3(a), it is seen that the concurrence C is increased first, then reached a maximum value, and finally decreased when Δ is increased from -0.4 to 1.0. For $\Delta > 0$, the concurrence C is monotonically decreased when T is increased. The concurrence C is plotted as functions of B and Δ in Fig. 3(b) when $T = 0.2$. From Fig. 3(b), it is seen that the value of C is kept zero for $\Delta \geq 0$ when $B = 0$. The peak in C appears at $\Delta = -0.5$. It is found that the anisotropy in the Heisenberg model can induce the entanglement between alternate qubits even for $B = 0$. If $B > 0$, the height of the peak in C is increased and the position of the peak is shifted to larger values of both B and Δ. From Fig. 3, it is seen that the curve of C is asymmetric about Δ.

In conclusion, in this report the entanglement between two alternate qubits of a four-qubit Heisenberg XX model is investigated. There is no thermal entanglement between alternate qubits of a four-qubit Heisenberg XX model when B is very small. However, when the magnetic field $|B|$ is increased, it can induce entanglement in the XX model both for the antiferromagnetic and ferromagnetic cases. The square wave like shape appeared in the concurrence C may be used as quantum entanglement switch. It is very interesting to find that the temperature affects the entanglement much stronger for weakly interacting alternate qubits than that for strongly interacting nearest qubits. The anisotropy in the Heisenberg model can also induce entanglement between alternate qubits.

Acknowledgement

It is a pleasure to thank Yinsheng Ling, Jianxing Fang, and Xiang Hao for their many helpful discussions and calculations.

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FIGURE CAPTIONS

Fig. 1.

The concurrence C is plotted as functions of magnetic field B and temperature T when $J = 1.0$.

(a). The curve of C as functions of B and T.

(b). The contour map of C as functions of B and T.

Fig. 2.

The concurrence C, the global entanglement Q, and the i-concurrence IC_i are plotted as a function of B when $J = 1.0$. For the concurrence C of (a) and (b), the curve is plotted when $T = 0.01, 0.1$, and 0.5 (from top to bottom).

(a). The curve C of the alternate qubits.

(b). The curve C of the nearest-neighbor qubits.

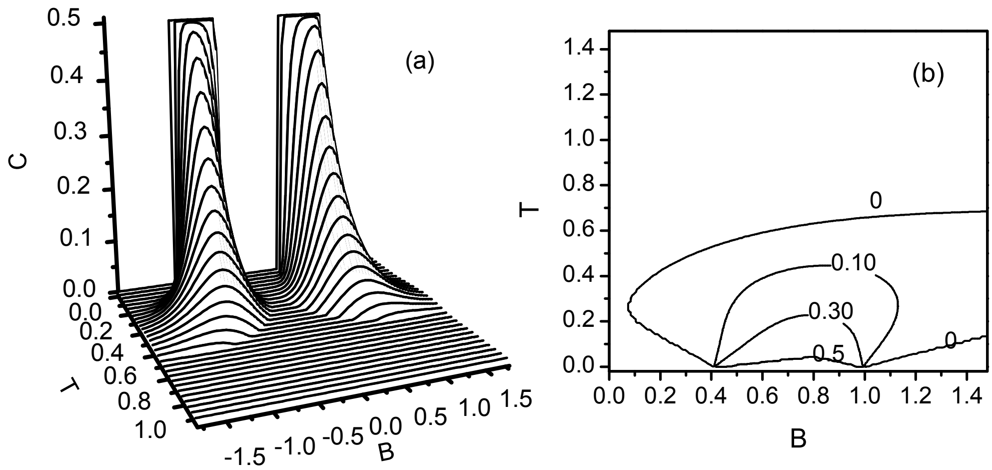
(c). The curves of Q and IC_i are plotted when $T = 0.01$.

—— : The curve of Q; - - - : The curve of IC_i.

Fig. 3. The concurrence C between alternate qubits is plotted as functions of T, B, and Δ when $J = 1.0$.

(a). The concurrence C as functions of T and Δ when $B = 0.5$.

(b). The concurrence C as functions of B and Δ when $T = 0.2$.



C
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
T
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
B
(a)
(b)
T
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
0.10
0.30
0.5
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
B

Fig. 1

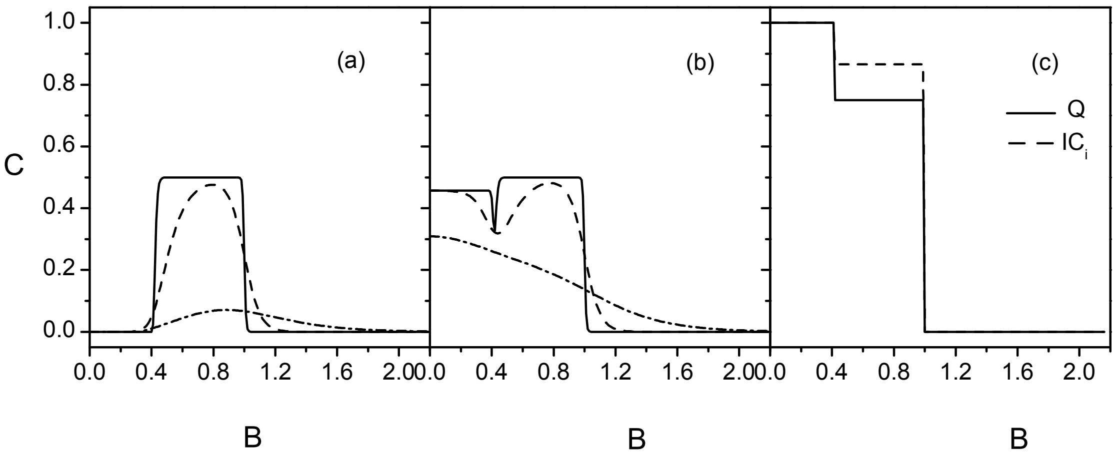
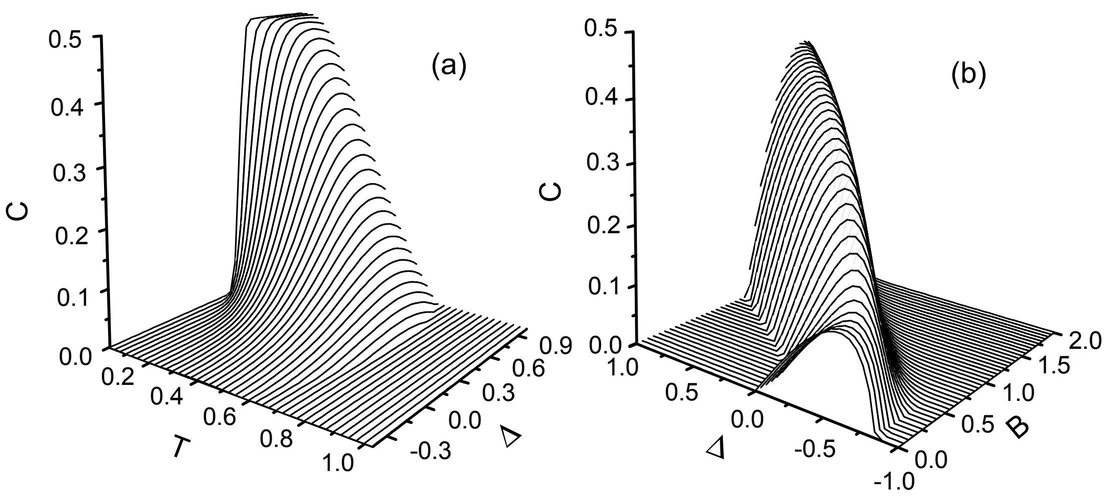


Fig. 2



(a)
C
0.5
0.4
0.3
0.2
0.1
0.0
0.2
0.4
0.6
0.8
1.0
T
-0.3
0.0
0.3
0.6
0.9
Δ
(b)
C
0.5
0.4
0.3
0.2
0.1
0.0
1.0
0.5
0.0
-0.5
-1.0
Δ
0.0
0.5
1.0
1.5
2.0
B

Fig. 3